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Evanescent coupling between discs: a model for near-integrable tunnelling

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Abstract

An analysis is provided of the splitting due to tunnelling of the energy levels in two-dimensional double-disk potentials. The formal setting of this problem is similar to that of tunnelling between tori in the near-integrable regime but is free of the difficulties arising from the existence of natural boundaries that are generic to such problems, enabling a systematic investigation of the relevant semiclassical theory. The semiclassical predictions are found to be consistent with exponentially accurate results obtained following reduction to a boundary-value problem. A numerical quantization is also performed and found to agree with the approximate results.

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1. Introduction

Recently, Pance *et al* [1] have measured the shifts in the microwave resonant frequencies of multiple dielectric discs that occur due to evanescent coupling between them. The theoretical analysis of this problem is similar to that for a quantum mechanical multiple-well potential in two dimensions, consisting of finite-step wells of circular shape. The resonant frequency shifts correspond to tunnelling splittings in the energy levels of the quantum mechanical problem. In this paper we give a theoretical analysis of tunnelling effects in this quantum-mechanical context and describe qualitative conclusions that can be reached for the experiment. This calculation provides a very clean application of a theory of tunnelling in near-integrable systems [2] which has previously been difficult to explore and test in detail.

A semiclassical calculation of the splittings places one in the regime considered by Wilkinson in [2], in which tunnelling rates between states supported on invariant tori of near-integrable systems are calculated. The results obtained by Wilkinson give a very elegant description of the tunnelling rate in terms of the geometry of the complexified tori in the region of complex phase space where they intersect. If two tori are assumed to intersect transversally in a section of complex phase space, the tunnelling rate between them is given in terms of an action connecting the two real tori and a symplectically invariant measure of the angle of intersection.

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Unfortunately, these appealing theoretical results are difficult to apply in practice. The reason is that, in generic KAM-like systems, an analytic continuation of the tori is defined only up to finite imaginary parts of the angle variables, where a natural boundary [3] is encountered beyond which analytic continuation of the tori is meaningless. This natural boundary is expected generically to be encountered *before* the intersection is reached [4]. Any application of the semiclassical result must be made in tandem with some form of regularization of the complexified tori in which classical dynamics is approximated in some way (and in fact there is no reason to believe that such a regularization even exists in general). Examples of applications to specific systems using various approximations of classical dynamics are to be found in [5–7].

Any application to a KAM-like system necessarily combines classical approximations with the semiclassical one and these are likely to strongly influence or even dominate the accuracy of the final result. In the multiple-disc problem, however, no such difficulty is encountered, allowing for the first time an implementation of Wilkinson's formula in which classical quantities are calculated exactly and without ambiguity. Direct analytic invariants (angular momenta about the disc centres) are available for the invariant tori within each disc and these allow straightforward analytic continuation as far into the forbidden regions as is necessary. While describing the real classical dynamics exactly within a single disc, these invariants are not global (in complexified dynamics) since the coupled problem does not have a rotational symmetry. The invariant tori defined by them therefore intersect nontangentially in the manner assumed by the Wilkinson calculation.

The geometry of the intersection can be completely characterized in terms of the disc radii and inter-disc separation and Wilkinson's formula for the splitting reduces to an explicit expression in terms of these parameters. This calculation is outlined in section 3 for the case of two identical discs, following an overview of the general theory of Wilkinson in section 2. In the subsequent section an exact calculation is presented based on methods developed in [9–11] for scattering from multiple-disc arrays. The results of the exact calculation are found to agree well with Wilkinson's approximation and furthermore allow an independent derivation of the semiclassical result in this context. Finally, we discuss qualitatively in section 5 the implications of these results for the measurements of Pance *et al* in [1].

2. Wilkinson's theory in quantum mechanics

We begin by summarizing Wilkinson's results as they apply to quantum mechanics in two dimensions, formulating them in such a way that their geometry is emphasized. We assume in this section that there are two families of congruent tori in phase space which may be semiclassically quantized to produce WKB quasimodes $\psi_{\mathrm{L}}(\boldsymbol{x})$ and $\psi_{\mathrm{R}}(\boldsymbol{x})$ such that the exact eigenfunctions may be approximated by

$$\psi_{\pm}(\boldsymbol{x}) = \frac{1}{\sqrt{2}}(\psi_{\mathrm{L}}(\boldsymbol{x}) \pm \psi_{\mathrm{R}}(\boldsymbol{x})). \tag{1}$$

The corresponding energy-level splitting may be calculated using Herring's formula [2]

$$\Delta E \approx \hbar^2 \int_{\Sigma} \left(\psi_{\mathrm{L}}^* \frac{\partial \psi_{\mathrm{R}}}{\partial n} - \frac{\partial \psi_{\mathrm{L}}^*}{\partial n} \psi_{\mathrm{R}} \right) \mathrm{d}\sigma \tag{2}$$

where Σ is any curve separating the two tori in configuration space. On substituting the WKB approximations for the quasimodes and performing the integral using the stationary-phase approximation we obtain Wilkinson's formula,

$$\Delta E \approx A(\hbar)\mathrm{e}^{-K_0/\hbar}. \tag{3}$$

The amplitude $A(\hbar)$ and exponent K_0 are obtained from the geometry of the invariant tori where they intersect in complex phase space, as we now explain. For simplicity we assume here that there is a single intersection branch governing the tunnelling rate (more generally one needs to sum over such intersections).

Denote by Λ_{L} and Λ_{R} the analytic continuations of the invariant tori to complex coordinates. In the forbidden region between the real tori, let λ_{L} and λ_{R} be the branches of these complexified tori from which the WKB quasimodes $\psi_{\mathrm{L}}(\boldsymbol{x})$ and $\psi_{\mathrm{R}}(\boldsymbol{x})$ are constructed. That is, λ_{L} and λ_{R} are the local branches of Λ_{L} and Λ_{R} such that integrating $\boldsymbol{p}\cdot \mathrm{d}\boldsymbol{x}$ along them gives actions $S_{\mathrm{L}}(\boldsymbol{x})$ and $S_{\mathrm{R}}(\boldsymbol{x})$ with positive imaginary part, corresponding to local WKB wavefunctions which decay into the forbidden region. Wilkinson's formula is evaluated in terms of the geometry of the intersection $\lambda_{\mathrm{L}}^* \bigcap \lambda_{\mathrm{R}}$. Note that, because λ_{L}^* and λ_{R} are each invariant under (complex) evolution in time, the intersection $\lambda_{\mathrm{L}}^* \bigcap \lambda_{\mathrm{R}}$ is expected generically to be a complex curve of two real dimensions, parametrizable by a single complex time coordinate.

The complex action K_0 is obtained by integrating $\boldsymbol{p}\cdot \mathrm{d}\boldsymbol{x}$ along a closed curve Γ on $\Lambda_{\mathrm{L}} \bigcup \Lambda_{\mathrm{R}}$, which goes from the local intersection $\lambda_{\mathrm{L}}^* \bigcap \lambda_{\mathrm{R}}$ along Λ_{L} to the conjugate local intersection $\lambda_{\mathrm{L}} \bigcap \lambda_{\mathrm{R}}^*$ and then returns along Λ_{R} to the starting point in $\lambda_{\mathrm{L}}^* \bigcap \lambda_{\mathrm{R}}$. Then

$$\oint_\Gamma \boldsymbol{p}\cdot \mathrm{d}\boldsymbol{x} = 2\mathrm{i}K_0$$

defines K_0. For the simplest topologies we consider, in which a single surface of intersection is relevant, K_0 is real.

The amplitude $A(\hbar)$ is a measure of the degree of nontransversality of the intersection. The local geometry of the surfaces λ_{L}^* and λ_{R} from which it is determined can be specified in two ways: one can specify vectors which span their tangent spaces or give functions for which they are simultaneous level sets. We begin with a specification of invariant functions. In two dimensions we expect that λ_{R}, for example, is a simultaneous level set of two functions, one of which should be the Hamiltonian if the surface is to be dynamically invariant. Let the second function be $I_{\mathrm{R}}(\boldsymbol{x},\boldsymbol{p})$ and let the corresponding function for λ_{L}^* be $I_{\mathrm{L}}(\boldsymbol{x},\boldsymbol{p})$. Then the amplitude in Wilkinson's formula can be written in the form [4],

$$A(\hbar) = 2\left(\frac{\hbar}{2\pi}\right)^{3/2}\sqrt{\frac{\omega_{\mathrm{L}}\omega_{\mathrm{R}}}{\mathrm{i}\{I_{\mathrm{R}}, I_{\mathrm{L}}\}}} \tag{4}$$

where ω_{L} and ω_{R} are frequencies of motion on the real tori and $\{I_{\mathrm{R}}, I_{\mathrm{L}}\}$ denotes a Poisson bracket, to be evaluated at any point on the intersection $\lambda_{\mathrm{L}}^* \bigcap \lambda_{\mathrm{R}}$. We note that the Jacobi identity gives $\{\{I_{\mathrm{R}}, I_{\mathrm{L}}\}, H\} = -\{\{I_{\mathrm{L}}, H\}, I_{\mathrm{R}}\} - \{\{H, I_{\mathrm{R}}\}, I_{\mathrm{L}}\} = 0$ and so $\{I_{\mathrm{R}}, I_{\mathrm{L}}\}$ is invariant under time dynamics and does not depend on the point in $\lambda_{\mathrm{L}}^* \bigcap \lambda_{\mathrm{R}}$ at which we evaluate it. The frequencies ω_{L} and ω_{R} are with respect to action variables which are conjugate to I_{L} and I_{R} respectively and we find that, for congruent tori such as we consider here, $\omega_{\mathrm{L}} = \omega_{\mathrm{R}}$.

The Poisson bracket may be reinterpreted in terms of the Hamiltonian flow vectors X_{L} and X_{R}, determined by I_{L} and I_{R} respectively, giving

$$\Xi_{\mathrm{LR}} \equiv \mathrm{i}\{I_{\mathrm{R}}, I_{\mathrm{L}}\} = \mathrm{i}\Omega(X_{\mathrm{R}}, X_{\mathrm{L}}) \tag{5}$$

where Ω is the symplectic form. In this form the interpretation of Ξ_{LR} in terms of the transversality of the intersection is more transparent—it is the area of a parallelogram formed by vectors tangent to each of λ_{L}^* and λ_{R}. In particular, note that in the integrable limit any invariants are global and λ_{L}^* and λ_{R} become tangent and so the amplitude diverges (in which case it should be replaced by a uniform approximation). In the case of a single contributing intersection as assumed here, Ξ_{LR} is a positive real number. Reality follows from the fact that any point on λ_{L}^* is mapped to a point on λ_{R} by a symmetry operation, σ say, which is

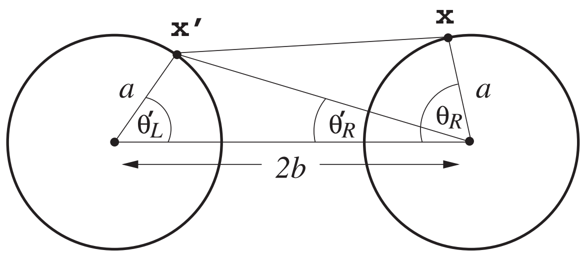

Figure 1. The coordinates used in describing the local solutions and in the expansion of the Green function. Note that the sense of positive θ_j changes from one disc to the other. The xy coordinates are centred at the midpoint between the discs, with the x-axis along the centre–centre axis.

symplectic. We find therefore that, as vector fields, X_{L} on λ_{L}^* maps under σ to X_{R} on λ_{R} and so $\Omega(X_{\mathrm{R}}, X_{\mathrm{L}}) = \Omega(\sigma X_{\mathrm{R}}, \sigma X_{\mathrm{L}}) = \Omega(X_{\mathrm{L}}, X_{\mathrm{R}})^* = -\Omega(X_{\mathrm{R}}, X_{\mathrm{L}})^*$, as claimed.

Finally, we note that, confined to a section as defined for example by $\boldsymbol{x} \in \Sigma$ and $H = E$, the intersection becomes an isolated point and the structure of the splitting is similar to a formula for the overlap of WKB wavefunctions given in [8]. As in that case, one finds in higher dimensions d that I_{L} and I_{R} become $(d-1)$-vectors of functions and the Poisson bracket term becomes a determinant [8] (and in addition the power 3/2 in (4) becomes $(d+1)/2$).

3. Application to the disc problem

We first apply the Wilkinson formula to the quantum mechanical problem of a double-well potential consisting of two identical disc-shaped wells of uniform depth $-V_0$ and radius a whose centres are a distance $2b$ apart. The corresponding microwave-dielectric problem has somewhat more complicated boundary conditions but shares the same general features. The quantum mechanical problem also has a simpler exact solution as outlined in the section following.

The exact solutions have a definite parity with respect to reflection in the y coordinate as well as in the x coordinate. We therefore use localized states of the form

$$\psi_j(\boldsymbol{x}) = \frac{1}{\sqrt{1+\epsilon^2}}[\psi_m(r_j, \theta_j) + \epsilon\psi_{-m}(r_j, \theta_j)] \tag{6}$$

where $j \in \{\mathrm{L}, \mathrm{R}\}$ and (r_j, θ_j) are polar coordinates centred around the respective disc centres oriented as in figure 1. We denote by $\psi_m(r, \theta)$ the single-disc solution with azimuthal quantum number m. When $m \neq 0$ we let $\epsilon = \pm 1$ designate the symmetry of $\psi_j(\boldsymbol{x})$ with respect to reflection in y and when $m = 0$ we take $\epsilon = 0$. We will also denote by $k(E) = \sqrt{2(V_0 + E)}$ and $\kappa(E) = \sqrt{-2E}$ the real and imaginary wavenumbers inside and outside the discs respectively. Henceforth we assume unit mass and set $\hbar = 1$.

When the localized states (6) are inserted in Herring's formula, four contributions are obtained, each of the form given in (3) and (4). There are identical contributions from the two products in which single-disc solutions occur with the same sign in m and a separate pair of identical contributions from those for which m occurs with opposing signs. We describe in detail below the contributions of the first kind and delegate detailed discussion of the second kind to the appendix.

With the conventions of figure 1, a pair of single-disc solutions with the same sign of m corresponds to quantized tori of whispering gallery type in which trajectories rotate in the clockwise direction in one disc and in the anticlockwise direction in the other. In the forbidden region between the discs, the angular momenta about the centres still take the real values $\pm m\hbar$ but the radial momenta are imaginary. One finds in particular an intersection of the complexified tori at a point in phase space whose configuration space component is the midpoint between the discs. From these midpoint intersections we obtain the following total contribution to the splitting:

$$\Delta E_{mm} = \mathcal{A}_m^2 \left[\frac{2}{(2\pi)^{3/2}} \frac{\omega_r}{\sqrt{\mathrm{i}\{L_\mathrm{R}, L_\mathrm{L}\}}} \right] \mathrm{e}^{-\Theta_m}. \tag{7}$$

The imaginary phase Θ_m depends on the separation between the discs and replaces $K_0/\hbar$. Using the notation $u = \kappa a$, $v = ka$ and $w = \kappa b$,

$$\Theta_m = 2(\xi(w, m) - \xi(u, m))$$

where

$$\xi(x, m) = \sqrt{m^2 + x^2} + m \ln \frac{x}{m + \sqrt{m^2 + x^2}}$$

is the exponent arising in the asymptotic form of the modified Bessel functions. It is easily verified that $\mathrm{i}\Theta_m/2$ is the action integral from the boundary of a disc to the midpoint between discs.

The term in square brackets in (7) is the amplitude given in (4). The role of the invariant functions I_L and I_R is played by the angular momenta L_L and L_R about the respective disc centres. One can show in general that for angular momenta L_1 and L_2 defined about different centres O_1 and O_2 with relative displacement vector $\boldsymbol{a} = \overrightarrow{\mathrm{O}_1\mathrm{O}_2}$, $\{L_1, L_2\} = \boldsymbol{a}\cdot\boldsymbol{p}$. Specializing to the case above and evaluating at the intersection between tori,

$$\mathrm{i}\{L_\mathrm{R}, L_\mathrm{L}\} = 2b\sqrt{\kappa^2 + m^2/b^2} = 2\sqrt{w^2 + m^2}.$$

Still in the square bracket, ω_r is the radial frequency of motion within the disc, corresponding to the radial action conjugate to the angular momentum. An explicit form for ω_r will not be required because it cancels in the final result.

The term $\mathcal{A}_m$ before the square bracket arises because the connection formula relating the wavefunction inside the disc to the solution in the forbidden region is not of the standard WKB type, since the potential changes discontinuously. It is defined implicitly by

$$\psi_m(\boldsymbol{x}) \sim \mathcal{A}_m \psi_{\mathrm{WKB}}(\boldsymbol{x}) \qquad \text{as} \quad r \to \infty$$

where $\psi_{\mathrm{WKB}}(\boldsymbol{x})$ is the solution with standard WKB normalization. Comparing the asymptotic form of

$$\psi_m(r, \theta) = \frac{K_m(\kappa r)}{K_m(\kappa a)} \frac{\mathrm{e}^{\mathrm{i}m\theta}}{\sqrt{2\pi a^2 N_m}} \qquad \text{for} \quad r > a \tag{8}$$

with the standard WKB form we find

$$\mathcal{A}_m = \frac{1}{\mathrm{e}^{\xi(u,m)} K_m(u)} \frac{\pi}{\sqrt{\omega_r a^2 N_m}}$$

where

$$N_m = \int_0^v \left[\frac{J_m(\eta)}{v J_m(v)} \right]^2 \eta \,\mathrm{d}\eta + \int_u^\infty \left[\frac{K_m(\xi)}{u K_m(u)} \right]^2 \xi \,\mathrm{d}\xi \tag{9}$$

ensures the normalization of $\psi_m(\boldsymbol{x})$.

Combining these conventions and definitions allows us to write (7) in the final form

$$\Delta E_{mm} = \frac{1}{a^2 N_m} \left[\frac{\sqrt{\pi}}{2} \frac{\mathrm{e}^{-2\xi(w,m)}}{(w^2 + m^2)^{1/4}} \right] \frac{1}{K_m(u)^2}. \tag{10}$$

Note that in our conventions $1/a^2$ is the fundamental quantum mechanical unit of energy. This result is to be added to the total contribution to the Herring integral from single-disc solutions with opposite signs in m (and pairs of tori with the same sense of rotation). It is shown in the appendix that these are of the form

$$\Delta E_{-m,m} = \frac{1}{a^2 N_m}\left[\frac{1}{2}\sqrt{\frac{\pi}{w}}\,\mathrm{e}^{-2w}\right]\frac{1}{K_m(u)^2}. \tag{11}$$

The final value for the fractional splitting is then

$$\Delta E \approx \Delta E_{mm} + \epsilon \Delta E_{-m,m}. \tag{12}$$

Note that a steepest-descents integration of Herring's formula requires only that the separation between discs be sufficiently large and in particular we do not require that $K_m(u)$ be in the asymptotic regime for these expressions to be valid. The formulas may hold for the ground-state splitting, for example.

4. Reduction to a boundary-value problem

An independent derivation of (10)–(12) and exact numerical solutions are obtained by adapting methods developed in [9, 10] for scattering from hard discs. These have been extended to the case of scattering from dielectric discs in [11]. In our case the calculation is somewhat simpler because we are interested solely in bound states and do not need to consider the scattering matrix in the whole energy plane. We will therefore modify the approach of [9–11] so that the calculation of the scattering matrix is bypassed. We first set up the basic method, following which we indicate how (10) may be obtained from it by perturbative methods and finally we compare the approximate results with numerically exact computation.

4.1. Derivation of the method

We seek a bound state $\psi(\boldsymbol{x})$. This must satisfy

$$(\nabla^2 + k^2)\psi(\boldsymbol{x}) = 0 \qquad \text{for} \quad \boldsymbol{x} \text{ in } D_{\mathrm{L}} \text{ or } D_{\mathrm{R}} \tag{13}$$

$$(\nabla^2 - \kappa^2)\psi(\boldsymbol{x}) = 0 \qquad \text{for} \quad \boldsymbol{x} \text{ outside } D_{\mathrm{L}} \text{ and } D_{\mathrm{R}} \tag{14}$$

and is subject to the boundary condition

$$\psi(\boldsymbol{x}) \to 0 \qquad \text{as} \quad |\boldsymbol{x}| \to \infty. \tag{15}$$

In the quantum mechanical problem, we have in addition the boundary conditions that the solution should be continuous and have continuous derivatives at the boundaries ∂D_{L} and ∂D_{R} of the discs D_{L} and D_{R}.

We consider expansions for $\psi(\boldsymbol{x})$ in three different regions of space appropriate to the three sets of boundary conditions imposed upon it. First, let (R, ϕ) be polar coordinates centred at some reference point in the neighbourhood of the discs. Since $\psi(\boldsymbol{x})$ must decay at infinity, we assume it can be expanded for sufficiently large R in the form

$$\psi(\boldsymbol{x}) = \sum_{m=-\infty}^{\infty} \gamma_m K_m(\kappa R)\mathrm{e}^{\mathrm{i}m\phi}. \tag{16}$$

Next, we take advantage of the continuity conditions on the boundaries of the discs. Let (r_j, θ_j), with $j \in \{L, R\}$, be polar coordinates centred at each of the discs. We assume that inside each disc an expansion of the form

$$\psi(\boldsymbol{x}) = \sum_{m=-\infty}^{\infty} \alpha_{jm} \frac{J_m(kr_j)}{J_m(v)}\mathrm{e}^{\mathrm{i}m\theta_j} \qquad \text{for} \quad \boldsymbol{x} \in D_j \tag{17}$$

is admissable. In particular, the continuity conditions imply that on the boundary ∂D_j (and just outside it) we may expand the wavefunction and its derivative in the forms

$$\psi(\boldsymbol{x}(\theta_j)) = \sum_{m=-\infty}^{\infty} \alpha_{jm} \mathrm{e}^{\mathrm{i}m\theta_j} \tag{18}$$

and

$$\frac{\partial\psi}{\partial n}(\boldsymbol{x}(\theta_j)) = -\frac{1}{a}\sum_{m=-\infty}^{\infty} \alpha_{jm} \frac{vJ_m'(v)}{J_m(v)} \mathrm{e}^{\mathrm{i}m\theta_j} \tag{19}$$

where the normal derivative is directed into the disc. We obtain quantization conditions by matching the expansions (16), (18) and (19) and demanding consistency between them.

This matching is achieved using the free-space Green function,

$$G(\boldsymbol{x}, \boldsymbol{x}') = -\frac{1}{2\pi} K_0(\kappa|\boldsymbol{x} - \boldsymbol{x}'|) \tag{20}$$

for the modified Helmholtz equation. Following [9–11], we let V be the region defined by $R \leqslant 1/\delta$ and $r_j \geqslant a + \delta$ and denote by $\partial_\infty V$ and $\partial_j V$ the parts of the boundary respectively corresponding to $R = 1/\delta$ and $r_j = a + \delta$. In the limit $\delta \to 0$ we obtain $\partial_j V \to \partial D_j$ but ∂D_j lies just outside V for small but finite δ. In particular, we obtain from the Green identity that, for $\boldsymbol{x} \in \partial D_j$,

$$\int_{\partial V} \left[G(\boldsymbol{x}, \boldsymbol{x}') \frac{\partial \psi(\boldsymbol{x}')}{\partial n'} - \frac{\partial G(\boldsymbol{x}, \boldsymbol{x}')}{\partial n'} \psi(\boldsymbol{x}') \right] \mathrm{d}\sigma' = 0. \tag{21}$$

As in the scattering case, there are three kinds of contribution to this integral: from $\partial_\infty V$, from $\partial_{j'} V$ with $j' \neq j$ and from $\partial_j V$.

In this bound-state calculation, substituting the expansion in (16) and performing the integral around $\partial_\infty V$ following expansion of the Green function using the Bessel-function addition theorems, we find no net contribution to the integral. We are therefore left with the contributions from $\partial_{j'} V$, where we must consider separately the cases $j = j'$ and $j \neq j'$. To simplify the calculation, we consider explicitly the case $\boldsymbol{x} \in \partial D_\mathrm{R}$ and adopt the convention for the angular coordinates θ_L and θ_R illustrated in figure 1. Following repeated expansion of the Green function using Bessel-function addition theorems we find in the limit $\delta \to 0$ that the contribution from $\partial_\mathrm{R} V$ is

$$I_\mathrm{RR} = \sum_{l,m=-\infty}^{\infty} \alpha_{\mathrm{R}m} D_{lm}(E)\, \mathrm{e}^{\mathrm{i}l\theta_\mathrm{R}}$$

where

$$D_{lm}(E) = I_l(u)\delta_{lm}\, K_m(u) \left[\frac{vJ_m'(v)}{J_m(v)} - \frac{uK_m'(u)}{K_m(u)} \right] \tag{22}$$

and that the contribution from $\partial_\mathrm{L} V$ is

$$I_\mathrm{RL} = \sum_{l,m=-\infty}^{\infty} \alpha_{\mathrm{L}m} C_{lm}(E) \mathrm{e}^{\mathrm{i}l\theta_\mathrm{R}}$$

where

$$C_{lm}(E) = I_l(u) K_{l+m}(2w) I_m(u) \left[\frac{vJ_m'(v)}{J_m(v)} - \frac{uI_m'(u)}{I_m(u)} \right]. \tag{23}$$

Contributions to the case $\boldsymbol{x} \in \partial D_\mathrm{L}$ are obtained by permuting the disc labels in these expressions.

The Green identity (21) evaluated on both boundaries then leads to the matrix equation

$$\begin{pmatrix} \boldsymbol{D} & \boldsymbol{C} \\ \boldsymbol{C} & \boldsymbol{D} \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha}_{\mathrm{L}} \\ \boldsymbol{\alpha}_{\mathrm{R}} \end{pmatrix} = \boldsymbol{0}.$$

In the limit $w = \kappa b \to \infty$ we find $\boldsymbol{C} \to 0$ and the solutions around the discs become decoupled. The solutions of the resulting equation

$$\boldsymbol{D}\boldsymbol{\alpha}_j = 0 \tag{24}$$

are obtained from the roots of

$$u J_m(v) K_m{}'(u) = v J_m{}'(v) K_m(u) \tag{25}$$

and these correspond to the solution for a single disc as expected.

4.2. Perturbative analysis

The solutions to the coupled problem can be projected onto the even and odd cases $\boldsymbol{\alpha}_{\mathrm{L}} = \pm\boldsymbol{\alpha}_{\mathrm{R}} = \boldsymbol{\alpha}$, which are respectively solutions of

$$(\boldsymbol{D} \pm \boldsymbol{C})\boldsymbol{\alpha} = \boldsymbol{0}. \tag{26}$$

This is easily solved by treating $\boldsymbol{C}$ perturbatively. First, however, it is convenient to rescale the matrices so that they take the following symmetric forms:

$$\tilde{D}_{lm}(E) = \delta_{lm} \left[\frac{u K_m'(u)/K_m(u) - v J_m'(v)/J_m(v)}{u I_m'(u)/I_m(u) - v J_m'(v)/J_m(v)} \right] \frac{K_m(u)}{I_m(u)} \tag{27}$$

and

$$\tilde{C}_{lm}(E) = K_{l+m}(2w). \tag{28}$$

Solutions of the uncoupled equation (24) are represented by vectors $(\boldsymbol{\alpha})_l = \delta_{lm}$ with E a root of (25). These are nondegenerate if $m = 0$ but are degenerate with $(\boldsymbol{\alpha})_l = \delta_{l,-m}$ if $m \neq 0$. Applying nondegenerate and degenerate perturbation theory respectively yields

$$\Delta E \approx \frac{2Q_{mm}(E)}{\tilde{D}'_{mm}(E)} \tag{29}$$

for the splitting at first order, where

$$Q_{mm}(E) = \tilde{C}_{mm}(E) + \epsilon \tilde{C}_{-m,m}(E) = K_{2m}(2w) + \epsilon K_0(2w) \tag{30}$$

and $\epsilon \in \{0, \pm 1\}$ is defined following the conventions described below (6). Because the numerator in the square bracket of (27) vanishes at the quantization condition (25), we have

$$\begin{aligned} \tilde{D}'_{mm}(E) &= \frac{\left[u K_m'(u)/K_m(u) - v J_m'(v)/J_m(v)\right]'}{u I_m'(u)/I_m(u) - u K_m'(u)/K_m(u)} \frac{K_m(u)}{I_m(u)} \\ &= \left[\frac{u K_m'(u)}{K_m(u)} - \frac{v J_m'(v)}{J_m(v)} \right]' K_m(u)^2 \end{aligned}$$

where the second line follows from the first on substitution of the Wronskian $W\{K_m(u), I_m(u)\} = 1/u$. By direct integration of (9) it is possible to show that the derivative term is simply

$$\left[\frac{u K_m'(u)}{K_m(u)} - \frac{v J_m'(v)}{J_m(v)} \right]' = 2a^2 N_m$$

and (29) can therefore be written in the form

$$\Delta E \approx \frac{1}{a^2 N_m} \frac{K_{2m}(2w) + \epsilon K_0(2w)}{K_m(u)^2}. \tag{31}$$

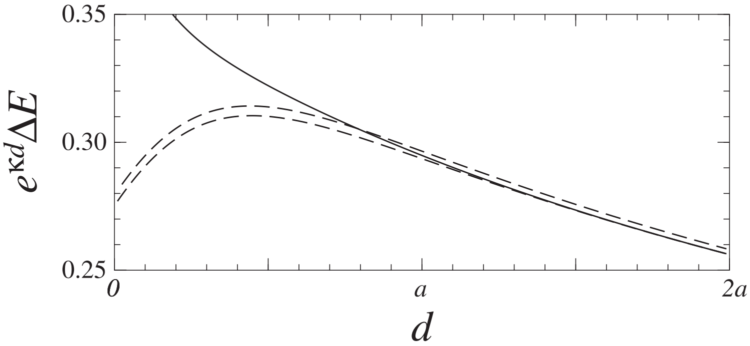

Figure 2. The solid curve gives the dependence of the scaled splitting $e^{\kappa d}\Delta E$ on the interdisc spacing d as determined from the zeros of $\Delta_\pm(E)$. The upper and lower dashed curves are respectively the fully semiclassical Wilkinson formula and the perturbative approximation (31). The splitting shown corresponds to the ground state of wells with $a = 1$ and $V_0 = 10$. Excited states exhibit qualitatively similar behaviour.

On substitution of the asymptotic form [12]

$$K_{2m}(2w) \sim \sqrt{\frac{\pi}{2}}\frac{e^{-\xi(2w,2m)}}{[(2w)^2 + (2m)^2]^{1/4}}$$

this reduces to Wilkinson's formula (10)–(12). Note that, while (10) is accurate to relative order $O(1/w)$ in the large parameter $w = \kappa b$, (31) is exponentially accurate since the only approximation has been a perturbative one in the exponentially small matrix $\boldsymbol{C}$.

4.3. Numerical solution

The original forms for $\boldsymbol{D}$ and $\boldsymbol{C}$ given in (22) and (23) are better suited to numerical analysis than the symmetric forms above because they are better behaved as $l, m \to \pm\infty$—we find, for fixed (u, v, w), that $D_{mm} \sim 1 + O(1/m^2)$ and $C_{lm} \to 0$ whereas elements of the symmetric forms diverge. In particular, the determinant

$$\Delta_\pm(E) = \det(\boldsymbol{D}(E) \pm \boldsymbol{C}(E))$$

exists and the energy levels may be determined from its zeros.

While the determinant converges relatively slowly with truncation dimension M, the roots themselves converge quite rapidly. For moderate values of the system parameters, convergence to machine precision can be achieved for $M \sim 10$ or less. We present some specific numerical results in figure 2. Only as the interdisc spacing $d = 2(b - a)$ approaches zero do we find a marked difference between the approximations we have developed and the exact splitting. Furthermore, the exponential accuracy of (31) for larger values of the spacing is confirmed. Although numerically less accurate, the raw semiclassical formula (10)–(12) captures the same qualitative features while incorporating the geometry of the problem in a more obvious way.

Note finally that variation with d of the scaled splitting $e^{\kappa d}\Delta E$ as seen in figure (2) is an intrinsically multi-dimensional feature. The analogous quantity in a one-dimensional problem would be constant.

5. Splittings of electromagnetic resonances

The methods outlined in the previous sections may also be applied to electromagnetic resonances of the kind reported in [1]. They are complicated by the fact that the relevant wave equation has a vector rather than a scalar character and the fact that derivatives may be discontinuous at the disc boundaries, but the same general conclusions are reached [13]. Ignoring dissipation, resonant frequencies of two identical dielectric discs placed in a microwave cavity exhibit a small splitting which varies with disc spacing in the same way as the energy splitting in (31). The only difference is that there is a different numerical constant in front (determined as before by the solution for a single disc).

In particular, this calculation predicts a frequency splitting which decays faster asymptotically than $e^{-\kappa d}$, where κ and d are respectively the decay constant in the evanescent region and the interdisc spacing (as before). In contrast, the frequency splittings reported in [1] decayed more slowly than $e^{-\kappa d}$. The most likely explanation for this seems to lie in the presence of dissipative effects, ignored in the present calculation. The slower decay of the experimentally measured splittings indicates that the finite lifetime of the states plays a dominant role there.

We note finally that, in the context of dielectric waveguides and resonators, there has been a longstanding interest in coupling between optical fibres [14]. Exponentially accurate estimates of coupling strengths similar to (31) are to be found in [15], for example. While possibly less accurate numerically, the Wilkinson approach has the advantage of incorporating the geometry of the problem in a more explicit and simple way. The approach outlined above might therefore have the advantage of being applicable to complex problems where the more accurate methods are difficult to apply.

6. Conclusion

We have verified that Wilkinson's formula gives a successful quantitative analysis of tunnelling effects in disc problems. The result is a simple prediction for this multi-dimensional tunnelling problem, which can be stated invariantly in terms of the symplectic geometry of the complexified tori. It agrees quantitatively with more accurate methods derived from reduction to a boundary value problem. The results, when compared to the experimental measurements of Pance *et al* [1], show a relatively more rapid decay of the resonant-frequency splitting, indicating that dissipation plays an important role in that experiment.

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Appendix. Contribution from the second intersection

The contribution to the splitting in (11) comes from the intersection of tori for which the angular momentum about each centre is the same. It is geometrically clear that any point in phase space lying on this intersection must have a momentum vector that is parallel to the axis joining the centres. In Cartesian coordinates we then have $p_y = 0$ and $p_x = -i\kappa$ (the sign on the latter being chosen to give a radially growing imaginary action for $\psi_R(\boldsymbol{x})$). The constraint that the clockwise angular momentum be m gives $y = -im/\kappa$. The intersection is then a two-real-dimensional complex line, which is naturally parametrized by the remaining (generally complex) coordinate x.

The Poisson bracket term evaluated on this intersection is

$$i\{L_R, L_L\} = 2ibp_x = 2\kappa b = 2w.$$

A straightforward but tedious calculation also allows us to confirm that an action integral along the respective torus branches starting at a point on the boundary of one disc, going to the intersection and on to the symmetric image on the second disc gives

$$\int \boldsymbol{p} \cdot \mathrm{d}\boldsymbol{x} = 2\mathrm{i}(\kappa b - \xi(\kappa a, m)).$$

From a similar sequence of manipulation to that outlined in section 3, we then obtain the partial contribution to the splitting given in (11).

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